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Power of One Bit of Quantum Information in Quantum Metrology

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We present a model of quantum metrology inspired by the computational model known as deterministic quantum computation with one quantum bit (DQC1). Using only one pure qubit together with l fully-mixed qubits we obtain measurement precision (defined as root-mean-square-error for the parameter being estimated) at the standard quantum limit, which is typically obtained using the same number of uncorrelated qubits in fully-pure states. In principle the standard quantum limit can be exceeded using an additional qubit which adds only a small amount of purity. We show that the discord in the final state vanishes only in the limit of attaining infinite precision for the parameter being estimated.

Emerging quantum technologies promise to solve problems that are intractable or impossible using classical counterparts. However, in many cases the origins of quantum enhancements remain the subject of debate. Entanglement unambiguously plays a critical role in many tasks that use pure states, but this often ceases to be true when noise is added to the picture [1]. One of the most studied tasks that uses noisy qubits is provided by a model called DQC1, introduced by Knill and Laflamme [2]. DQC1 performs efficiently a specific type of computation using highly-mixed quantum states which is thought to be hard classically, and thereby seriously challenges the notion that pure-state entanglement plays an essential role in quantum computation.

The task performed by DQC1 is to estimate the normalised trace of a quantum circuit U that acts on a collection of l register qubits, as depicted in Fig. 1(a). The initial state comprises one “clean” pure qubit together with register qubits that are maximally mixed, and only unitary gates are used for the computation. Remarkably, the precision of the estimate does not scale with the size of U . It is intuitively clear that DQC1 achieves an exponential speedup over any known classical algorithm for several applications [3–7], and it is widely believed that classical simulation of DQC1 is hard [8]. Several works have also analysed how the computational power of DQC1 changes as resources, such as additional pure qubits and measurements, are added [5, 6, 8], see Fig. 1(b).

Some studies have also investigated the role of entanglement and quantum discord [9, 10] in the speedup achieved by DQC1 [11] by looking at correlations generated at the output [12, 13]. It was shown that the discord generated by Haar-random unitary circuits remains a fixed proportion of the maximum possible as the circuits increase in size, while the amount of entanglement generated is vanishing. These results have prompted widespread interest in the hypothesis that the generation of discord by DQC1 circuits plays a critical role in the exponential speedup which is achievable for estimating normalized trace (compared to all known classical algorithms). However, this hypothesis remains unproven, and furthermore it is not yet known what happens to entanglement or discord at intermediate steps in DQC1 computations.

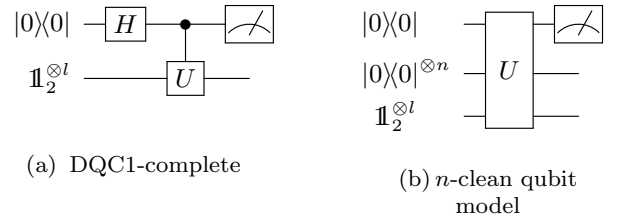


FIG. 1. (a) A DQC1-complete problem is to compute the normalised trace of a unitary transformation. After several runs of the circuit, an estimate is obtained for $\langle \sigma_x \rangle + i \langle \sigma_y \rangle = \text{tr}(U)/2^l$, with precision depending only on the number of runs. The protocol also works when the control qubit is partially pure at the start – as given by the state in Eq. (5). In this case, the number of runs must be increased by a factor $1/\epsilon^2$ to achieve the same precision as when the control qubit is initially pure. (b) For the general DQC1 problem, an additional $n \sim \log(l)$ pure qubits can be introduced without altering the computational power [5, 6].

We now turn to quantum metrology, and achieving quantum advantage for precision in the task of phase estimation, which is used for highly-sensitive measurements of physical parameters [14–16]. Phase-estimation strategies that cannot exploit quantum features are subject to the *standard quantum limit* (SQL) for precision, given by $\Delta\phi = 1/\sqrt{n}$ where n particles are used for the probe, ϕ is the parameter to be estimated, and $\Delta\phi$ is the root-mean-square-error for estimates of ϕ . For example, this limit applies when n particles in the $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ state are used to measure the phase for a Pauli rotation

$$u_\phi = e^{i\phi\mathbf{g}} \quad \text{where} \quad \mathbf{g} = |1\rangle\langle 1|. \quad (1)$$

However, when a GHZ state $|+_n\rangle = (|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$ is used as the probe state with $G = \sum_{j=1}^n \mathbf{g}_j$, the precision scales at the Heisenberg limit $\Delta\phi = 1/n$, which is the best precision achievable [15].

For mixed-state models of phase estimation, recent results challenge any presupposed link between entanglement and

quantum advantage for measurement precision. Ref. [17] considers an algorithm for multi-parameter estimation using DQC1. This algorithm uses an adaptive protocol based on a series of estimates with different interaction times, to achieve a final precision scaling with the inverse total interaction time. Ref. [18] analyses the situation where a unitary circuit is used to prepare probe states from n uncorrelated qubits in the state ρ_ϵ given below in Eq. (5). It was found that circuits which generate non-classical correlations can achieve a quadratic quantum advantage compared to circuits generating only classical correlations at fixed ϵ . This result holds even for small values of ϵ where there is no entanglement but large amounts of discord, and the amount of discord also grows with n . Another recent analysis considers phase estimation using an interferometer, where the spectrum of the interferometer Hamiltonian is fixed but not its eigenbasis [19].

Inspired by DQC1, we now ask whether a large ensemble of mixed qubits can be used as the basis of a powerful sensor. We consider a model where only one (or few) clean qubits are accessible, and only one qubit can be measured at the end [20]. Physical systems where our model applies include nuclear magnetic resonance [21–23] and some cold-atom systems [24]. Ordinarily for these systems only bulk operations on the register qubits are available — which is to say the same operation is applied to every register qubit, optionally under global control. Hence, we demand that only bulk operations are permitted for preparation the probe state and for the readout procedure.

Parameter estimation: In the theory for parameter estimation [25] a process alters an initial distribution \mathbf{p} into $\mathbf{p}(\phi)$, which is a function of a single parameter ϕ . The value of ϕ can be determined by differentiating between the initial and the final distributions. The uncertainty in this value is bounded by the Fisher information F , which is given by:

$$\Delta\phi \geq \frac{1}{\sqrt{F}}, \quad \text{with} \quad F = \sum_k \frac{[\partial_\phi p_k(\phi)]^2}{p_k(\phi)} \quad (2)$$

and p_k is the probability for observing outcome k . The above inequality is the Cramér-Rao bound [26, 27].

When using a quantum system, the initial and final probability distributions are replaced by density operators ρ and $\rho(\phi)$ respectively. The final state is measured by a positive-operator-valued measure (POVM) $\{\Pi_k\}$ to yield classical probabilities $p_k = \text{tr}[\Pi_k \rho(\phi)]$, from which F in Eq. (2) can be computed. If the process which is parameterized by ϕ is unitary then F , when optimised over all POVMs, is given by the quantum Fisher information [28]:

$$F_Q = 4 \sum_{i>j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |\langle \psi_i | G | \psi_j \rangle|^2, \quad (3)$$

where $\{\lambda_i\}$ are the eigenvalues, $\{|\psi_i\rangle\}$ are the eigenvectors of ρ , and G is the Hamiltonian generator of the phase shift. This formula for F_Q yields a tight lower bound for precision

without needing the explicit form of an optimal POVM, and enables a straightforward comparison between probe states.

The setup: Our model uses three registers: one with n pure qubits; one with m qubits with finite purity as given in Eq. (5); and one with l fully-mixed qubits. Along with these three registers, there is one pure qubit in state $|0\rangle$ which serves as the control. The total initial state is

$$\varrho_0 = |0\rangle\langle 0| \otimes |0\rangle\langle 0|^{\otimes n} \otimes \rho_\epsilon^{\otimes m} \otimes \frac{\mathbb{1}^{\otimes l}}{2^l}, \quad (4)$$

$$\text{where} \quad \rho_\epsilon = \frac{1}{2} \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 1-\epsilon \end{pmatrix}, \quad 0 < \epsilon < 1. \quad (5)$$

To prepare the probe state ϱ we apply the Hadamard gate to the control qubit followed by a CNOT gate for all qubits in the register. Next each qubit in the register is allowed to evolve freely under the unitary operation given in Eq. (1). The readout procedure consists of another controlled operation and measurement of the control qubit. The full protocol is shown in Fig. 2.

To compute F_Q for ϱ above, we note that ϱ_0 has eigenvectors of the form $|\pm; b_0^n; b_j^m; b_k^l\rangle$; here b_i^a represents a binary string of length a with 1s appearing i times, and the semicolons separate the control qubit and the three registers. There are $\binom{m}{j} \binom{l}{k}$ such eigenvectors each with eigenvalue $\lambda_j^+ = \frac{1}{2^{m+l}}(1+\epsilon)^{m-j}(1-\epsilon)^j$ when the control qubit is in state $|+\rangle$, and the eigenvalue is $\lambda_j^- = 0$ otherwise. After the first CNOT gate the eigenvectors are $|\psi_{jk}^\pm\rangle = (|0; b_0^n; b_j^m; b_k^l\rangle \pm |1; c_0^n; c_j^m; c_k^l\rangle) / \sqrt{2}$, where c_j^a is the NOT of b_j^a , i.e., $|c_j^a\rangle = \sigma_x^{\otimes a} |b_j^a\rangle$.

The generator of the phase shift is $G = \sum_s |1\rangle\langle 1|_s \otimes \mathbb{1}_{\bar{s}}$, where $\mathbb{1}_{\bar{s}}$ is the identity operator on all but s th qubit, and s runs from 1 to $n+m+l$. Next, we note that the components of the eigenstate of the prepared state are eigenvectors G :

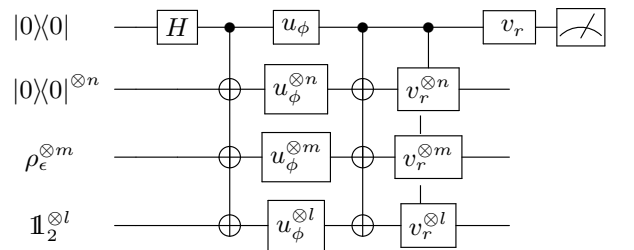


FIG. 2. General scheme for phase estimation: There are n pure qubits, m semi-pure qubits, and l fully-mixed qubits ($\mathbb{1}_2 = \mathbb{1}/2$). The first qubit is the control, and the remaining qubits constitute the register. A bulk CNOT is used to prepare the probe, and each qubit is then subjected to the unitary operation given in Eq. (1) for which ϕ is to be determined. The readout procedure is adaptive: θ_r is the estimate for ϕ after the first $r-1$ rounds. This value is used to configure the readout circuit, which is a bulk CNOT followed by a bulk controlled- v_r , and measurement on the control.

$$G|0; b_0^n; b_j^m; b_k^l\rangle = (j+k)|0; b_0^n; b_j^m; b_k^l\rangle, \quad G|1; c_0^n; c_j^m; c_k^l\rangle = (n+1+m-j+l-k)|1; c_0^n; c_j^m; c_k^l\rangle \quad (6)$$

$$\text{and therefore } \langle \psi_{j',k'}^\pm | G | \psi_{j,k}^\pm \rangle = \frac{1}{2}(j+k)\langle b_{j'}^m | b_j^m \rangle \langle b_{k'}^l | b_k^l \rangle \pm \frac{1}{2}(n+m+l-j-k+1)\langle c_{j'}^m | c_j^m \rangle \langle c_{k'}^l | c_k^l \rangle. \quad (7)$$

Eq. (7) is non-zero only when $j = j'$ and $k = k'$. We note that the numerator of the first term in Eq. (3) is the difference in two eigenvalues, and therefore it is only necessary to consider $\langle \psi_{j,k}^- | G | \psi_{j,k}^+ \rangle$. Hence,

$$F_Q = 4 \sum_{j=0}^m \binom{m}{j} \lambda_j^+ \sum_{k=0}^l \binom{l}{k} \left| \langle \psi_{j,k}^- | G | \psi_{j,k}^+ \rangle \right|^2 \\ = l + m(1 - \epsilon^2) + (1 + n + \epsilon m)^2. \quad (8)$$

We can now make several observations: (i) F_Q is always greater or equal to the SQL value, which is $1 + l + m + n$. (ii) The SQL is attained when $m = n = 0$, i.e. the case which is analogous to DQC1 [29]. (iii) If ϵ is small (or even 0) there is a linear contribution of m corresponding to size of the register of partially-pure qubits. (iv) $(n+1)^2$ exhibits the well-known quadratic enhancement for entangled pure states of $n+1$ qubits, and there is an additional contribution equivalent to ϵm extra pure qubits. (v) Our protocol achieves optimal precision under the assumptions of the standard DQC1 input state ($m = n = 0$), access to controlled unitary operations for preparation and measurement steps, and identical unknown phase rotations on all qubits, see the Appendix A.

Finally, for $m > 1$ the purity from the partially-pure qubits can be concentrated using into a single qubit using a purity distillation protocol [30]. Doing this will increase the number of both fully pure and fully mixed qubits. It would be interesting to compare the performance of such a device. However, we will address such cases in detail in a future manuscript.

Readout procedure: Next we consider how F_Q given in Eq. (8) can be attained via a suitable POVM, which in general can require entangled measurements [31]. For our model, attention must be given to the bulk-operation requirements for implementing the measurements for the readout procedure, and the following method suffices (illustrated in Fig. 2): a bulk CNOT gate is performed, followed by a bulk controlled- v_r where $v_r = \exp\{-i\theta_r \sigma_z\}$, and a measurement on the control qubit. θ_r here is taken to be the estimate of ϕ after $r-1$ rounds. The initial estimate θ_0 can assume no prior knowledge of ϕ . In each successive round our estimate for ϕ is improved, i.e., $|\theta_r - \phi| < |\theta_{r-1} - \phi|$, using an adaptive Bayesian update or maximum-likelihood method to maximize sensitivity [32].

The measurement of the control qubit along the σ_x direction yields probability distribution

$$q_r^\pm = \frac{1}{2}(1 \pm x_r) \quad \text{where} \quad (9)$$

$$x_r = \text{Re} \left[e^{i(n+1)\omega_r} \cos^l(\omega_r) \{ \cos(\omega_r) + i\epsilon \sin(\omega_r) \}^m \right],$$

and $\omega_r = \theta_r - \phi$ (see details in Appendix B). The value for F computed from this probability distribution, using Eq. (2),

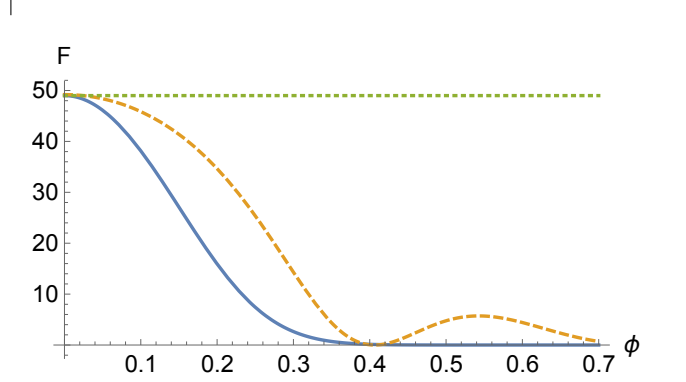


FIG. 3. (Colour online.) F computed from the probability distribution in Eq. (9) for $n = 6$ and $m = l = 0$ (dotted); $n = l = 0$ and $m = 11$ with $\epsilon = 0.49$ (dashed); and $l = 48$ and $n = m = 0$ (solid). All three cases have the same value for F_Q . When there is no prior information about the value of ϕ , an initial estimate of its value can be obtained using a small number of measurements with single-qubit probes, half with an adaptive setting $\theta = 0$ and half with $\theta = \pi/2$. A high-precision estimate can then be obtained using a larger number of measurements with probe states having register qubits. The adaptive procedure used at this stage would adjust the setup towards maximum sensitivity, while ensuring that estimates of ϕ do not become multi-valued.

yields a value that approaches the quantum Fisher information in Eq. (8) as $\omega_r \approx 0$. That means that the adaptive protocol described above will yield the optimal Fisher information as the estimate θ_r approaches ϕ . We have plotted three cases in Fig. 3.

The features of our metrology protocol can now be related to the general DQC1 problem Fig. 1(b), provided that the total number of (partially)-pure qubits $n + m$ scales as $\log(l)$ (as is assumed for DQC1 algorithms). For operations involving the register in Fig. 2, there is no change to the initial state of the register when the control is in $|0\rangle$; when the control is in state $|1\rangle$, the unitary operation is

$$U_r = \left(u_\phi^\dagger v_r \sigma_x u_\phi \sigma_x \right)^{\otimes l+m+n}. \quad (10)$$

For the purposes of comparison with known results, additional rotations on the control in Fig. 2 are not relevant, and we can consider the circuit as an application of a bulk controlled-unitary operation ${}^c U_r$. With a view to shedding new light on the discord hypothesis for DQC1 [13], we will examine the role of correlations (discord and entanglement) in our metrology protocol, both for the probe and output states.

One+ ϵ clean qubit metrology: Let us first consider the case where $n = 0$, $m = 1$, and $l > 0$. In this case the probe state ϱ is entangled for any value of $\epsilon > 0$. This can be understood by

noting that a non-positive partial transpose for ϱ of the state results from applying a CNOT gate on the state in Eq. (5), controlled on $|+\rangle$. Another way to see this is by noting that the value for F_Q here beats the SQL [33]:

$$F_Q = l + 2 + 2\epsilon > l + 2. \quad (11)$$

In other words, even with one qubit with finite purity we can attain better precision than what is achievable using the same number of pure probe qubits which probe the field independently (with phase encoded once onto each qubit). Adding more qubits to the registers for initially partially-mixed and pure qubits, the entanglement (between the control and registers) will increase as well as the value for F_Q .

One-pure-qubit metrology: We now let $n = m = 0$, i.e., consider a $l + 1$ qubit state with only one pure qubit and l qubits in fully-mixed state. From Eq. (8) we see that F_Q has the SQL value of $l + 1$ qubits, and that the SQL is attained using only one pure qubit and l fully-mixed qubits. This is highly counterintuitive in the classical setting for which only uncorrelated qubits are used as probes. Here completely-mixed states cannot be used to yield additional information from a phase measurement, and the maximum value for F would be 1 (as is attained using a single pure qubit). Therefore the enhancement of F_Q by l is fundamentally quantum.

It is tempting to say that the resource enabling this enhancement in F_Q is the entanglement or discord in ϱ . However, a closer look at ϱ in the limit $\epsilon \rightarrow 0$ reveals that it is an equal mixture of products of eigenstates of σ_x (for which $|-\rangle$ occurs even number of times),

$$\varrho = \frac{1}{2^{l+1}} \left(\mathbb{1}^{\otimes l+1} + \sigma_x^{\otimes l+1} \right), \quad (12)$$

and it is therefore fully classically correlated [34]. Though ϱ is separable, and therefore preparable via unrestricted LOCC, it cannot be prepared using bulk LOCC operation. Without the CNOT gate used in the state preparation, which is controlled on a quantum superposition, the register of maximally-mixed qubits cannot be exploited.

At this point we can ask whether there is any discord present in the final state of the circuit. In Ref. [35] it was shown that there is no discord in the output state of a DQC1 circuit when the controlled-unitary operation is Hermitian, i.e. $U = U^\dagger$ in Fig. 1 (see also Refs. [36] and for further details). The unitary operator U_r , in Eq. (10), is Hermitian if and only if $\omega_r = 0$, i.e. when ϕ is known to perfect precision. Therefore it may be observed that the circuit in Fig. 2 contains discord for all runs except when ϕ is fully known. Repeating this analysis for arbitrary values of $l, m, n > 0$ shows that the final state is always separable, but has finite discord except when $\omega_r = 0$. The only exception is when $l = m = 0$, in which case the final state has no correlations. We may conclude that noisy input states lead to discordant output states in our model, which sheds new light on the constant level of discord at the output of DQC1 found in Ref. [13].

Discussion: Our results provide support for both entanglement and discord as enabling quantum resources in quan-

tum metrology. They are complementary to those reported in Ref. [37], which showed that noisy multi-qubit states containing only bound entanglement can be used as probe states for phase estimation, for which F_Q scales quadratically with the number of qubits (corresponding to scaling at the Heisenberg limit up to a constant factor). They also contribute to efforts to find and exploit relationships between quantum advantages in metrology and speedups in quantum algorithms, such as was studied in Ref. [38] with regards to query complexity for quantum search problems.

Perhaps more importantly, our model shows how a large ensemble of highly-mixed quantum systems can be of great utility for quantum sensing. Since our model only requires bulk coherent operations on the ensemble, it has the potential to enable a scalable quantum technology which could challenge state-of-the-art classical sensors in the near future [39]. The biggest practical weakness of our model lies in the fact that all sensitivity vanishes if qubits are lost (and even just one of them) between the first and last controlled gates — a problem which is shared by any measurement device using pure GHZ states or NOON states in the context of interferometry [14]. An experimental setup which is well suited to implement our protocol is provided by ultra-cold atoms in adjacent optical dipole traps, where controlled gates on a large number of register atoms can be performed using an approach proposed in Ref. [24] using electromagnetically-induced transparency (EIT) and Rydberg blockade. A detailed proposal is provided in Ref. [39], with applications including ultra-precise measurements of gravity.

Appendix A.— For the case that the register is fully mixed at the start, the bulk constraint dictates that classical correlations for the probe state must be generated without a “classical” strategy, i.e., local operations and a probabilistic method, and it can only be prepared via a coherent quantum interaction. Moreover, phase encoding must satisfy the same constraint, i.e., $U_\phi = u_\phi^{\otimes n}$. Otherwise the pure control qubit can be swapped around so that the phase is encoded sequentially onto the pure qubit, which achieves Heisenberg limit precision [40].

We may think of our model as application of three unitary operations on all qubits: (i) a preparation U_P ; (ii) the phase encoding unitary U_ϕ ; (iii) a readout unitary U_R . Let us denote $U_T = U_R U_\phi U_P$. If we restrict U_P and U_R to be controlled unitary operations, then σ_x measurements on the controlled qubit yields probabilities $p_\phi(\pm) = (1 \pm \frac{1}{2} \text{tr}[U_R U_\phi U_P]) / 2$. By cyclicity of trace we can put U_P and U_R together: $U_\omega = U_P U_R$. Now, it’s clear via convexity of Fisher information that we want to maximise the overlap in the basis of U_ϕ and U_ω . This can be done by ensuring that the two unitary transformations have the same eigenbasis. And, that is precisely what our protocol does. It puts U_ω in the same form as U_ϕ , and then adaptively chooses values for ω such that $\omega \rightarrow \phi$. This makes our protocol optimal.

Appendix B.— After applying the Hadamard gate, CNOT gate, followed by encoding the phase [41], and another CNOT gate, we have

$$\varrho_f = \frac{1}{2^{l+1}} \left(\rho_\epsilon^{\otimes m} \otimes \mathbb{1}^{\otimes l} e^{-i(n+1)\phi} (\sigma_x u_\phi \sigma_x \rho_\epsilon u_\phi^\dagger)^{\otimes m} \otimes (\sigma_x u_\phi \sigma_x u_\phi^\dagger)^{\otimes l} \right) \otimes |0\rangle\langle 0|^{\otimes n}.$$

Lastly, we apply controlled- v_{θ_r} as well as v_{θ_r} on the control qubit yielding the final state ϱ_f . Note that v_{θ_r} also commutes with $|0\rangle\langle 0|$, ρ_ϵ , and $\sigma_x \rho_\epsilon \sigma_x$. Measuring the control qubit in the basis of σ_x gives us Eq. (9).

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